RESEARCH PROPOSAL

RHODES UNIVERSITY

EDUCATION DEPARTMENT

(Mathematics Education)

Candidate: Beata Lididimikeni Dongwi

Student Number: 609D6388

Degree: PhD

Provisional Title

Examining mathematical reasoning through enacted visualisation when solving word problems

Supervisor: Prof. Marc Schäfer

Abstract

The need to study mathematical reasoning as an essential tool in problem solving in secondary school mathematics is fairly widespread in recent research studies in mathematics education. This qualitative case study aims to develop and analyse visualisation/reasoning tasks called the Enacted Visualisation Geometric Reasoning Tasks (EVGRTs) that will be solved by the selected Grade 11 learners. They will be analysed to examine and interrogate how visualisation processes enhance reasoning processes when solving geometry word problems. The study also aims to analyse how enacted visualisation processes co-emerge with mathematical reasoning whilst solving geometry word problems. The study is framed by an enactivist perspective and it is underpinned by an interpretive paradigm. The sample comprises a cohort of purposefully selected Grade 11 learners who will individually and collaboratively solve the EVGRTs. Data will be collected and analysed in two phases. The analysis involves observation of small-groups collaborative argumentation sessions, small-groups interviews and focus group interviews with the whole cohort. It is hoped that the study will contribute towards understanding the links between visualisation, reasoning and enactivism as these are interesting and have not attracted much attention both empirically and theoretically in the context of mathematics education.

Keywords
Collaborative argumentation, enactivism, enacted visualisation, geometry word problems, mathematical reasoning, visualisation processes, visualisation.

Common Statement:

This proposed research study is part of the 'Visualisation in Namibia and Zambia" (VISNAMZA) project which seeks to research the effective use of visualisation processes in the mathematics classrooms in Namibia and Zambia (Schäfer, 2015). Research in the VISNAMZA project is currently centred on 5 MEd studies and 1 PhD study.

FIELD OF RESEARCH

Visualisation and reasoning in Mathematics Education

PROVISIONAL TITLE

Examining mathematical reasoning through enacted visualisation when solving word problems

CONTEXT

INTRODUCTION

The need to study mathematical reasoning as an essential tool in problem solving in secondary school mathematics is fairly widespread in recent research studies in mathematics education (Boesen, Lithner, & Palm, 2010; Dejarnette & González, 2013; Mueller, Yankelewitz, & Maher, 2014). Russell (1999) defines mathematics as "a discipline that deals with abstract entities and reasoning is the tool to understand abstraction" (p. 1). Lithner (2000) defines reasoning as "the line of thought, the way of thinking, adopted to produce assertions and reach conclusions" (p. 166).

Malloy (1999) proposes the use of "mathematical questioning (or inquiry) by students and teachers as a strategy to help students use their innate reasoning abilities to sharpen and clarify their understanding of mathematical concepts" (p. 13). English (1999) asserts that reasoning by analogy is also an important tool in problem solving and problem posing (p. 35) as it "entails understanding something new by comparison with something that is known" (p. 22). English (1999) further states that learners do not reason by analogy if they do not see the "connections and relationships among mathematical ideas and using these understandings to master new situations" (pp. 22–23).

Word problem solving is one of the significant aspects of mathematical problem solving which incorporates actual problems and mathematical applications. However, research reveals that learners "express great difficulties in handling a word or story problem" (Ahmad, Tarmizi, & Nawawi, 2010, p. 356). Drawing a diagram is a common suggested strategy for solving word (story) problems in mathematics (Stylianou, 2002; Ahmad et al., 2010). In their study, David & Tomaz (2012) present "an illustrative episode that shows how drawings of geometrical figures have a powerful role in structuring and modifying mathematical activity in the classroom" (p. 413). Chen & Herbst (2013) add that the nature of diagrams could play an important role in students' geometrical reasoning and consequently help them to make reasoned conjectures (p. 304). Drawings are important elements in the use of multiple representations to support abstraction (Jao, 2013).

When solving mathematical problems, learners go through different steps, methods and procedures to explain, justify, argue and generalize their solutions and problem solving strategies. Visualisation is one of the methods that learners can employ during problem solving. Zimmermann and Cunningham (1991) define mathematical visualisation as “the process of producing or using geometrical or graphical representations of mathematical concepts, principles or problems, whether hand drawn or computer generated” (p. 1). The purpose of this study is to examine the role of visualisation in enacting selected Grade 11 learners' mathematical reasoning when solving geometry word problems (GWPs). Conceptually, the study foregrounds three main constructs – mathematical reasoning, visualisation and word problems as discussed below.

MATHEMATICAL REASONING

One of the aims of the mathematics curriculum in Namibia is to enable learners to use mathematics as a means of communication with an emphasis on the use of clear expressions and to “develop the abilities to reason logically, to classify, to generalise and to prove” when presented with real life situations (Namibia. MoE, 2010, p. 2). Stein, Grover and Henningsen (1996) promote the use of task features which support high cognitive demands on learners, including reasoning and sense-making. These features are “the existence of multiple-solution strategies, the extent to which the task lends itself to multiple representations and the extent to which the task demands explanations and/or justifications from the students” (p. 461). Brodie (2010) adds that “tasks that support multiple voices, disagreements, and challenges also support mathematical reasoning, when used appropriately” (p. 7).

Brodie (2010) views mathematical reasoning as a means to sense-making of and in a mathematical activity. She assumes that “only through making sense of the mathematics can we truly move to sense-making as a worthwhile everyday life activity” (p. 59). Bjuland (2007) perceives sense-making, conjecturing, convincing, reflecting and generalizing as interrelated processes of mathematical thinking and reasoning (p. 2). Brodie (2010) affirms that “mathematical reasoning is what mathematicians do – it involves forming and communicating a path between one idea or concept and the next” (p. v).

Children are naturally curious and learn to make sense of their world through exploration, questioning and reasoning. As they grow older, their questioning strategies and reasoning skills shift to suit the purpose of the phenomenon under study. Like many mathematics teachers and researchers in the field of mathematics education, I am concerned with the learners' difficulties to express their solutions to GWPs. There often seems to be a gap between the learners' solution to such problems and their ability to explain how they obtained that solution. They find it very difficult to talk about their problem solving strategies irrespective of the accuracy of their methods and/or solutions. In my experience, they find it very difficult to make their reasoning explicit.

Sternberg (1999) urges that "one cannot solve a problem until one identifies the nature of the problem to be solved" (p. 41). Hence, having figured out the nature of the problem, the learner needs to "figure out a strategy that will effectively solve it" (*ibid*.). Thereafter, he/she needs to represent the problem using a multitude of visual representations and evaluate his/her work during and after problem solving to ensure that solutions make sense and are error free. He further emphasizes that "a good assessment of mathematical reasoning will evaluate all [the five mentioned] aspects of mathematical reasoning" (*ibid*., p. 43).

I concur with Sternberg (1999) that at school we often create a closed system "that consistently rewards students who are skilled in memory and analytical abilities ... but fails to reward students who are skilled in creative and practical abilities" (p. 38). It is no wonder students who seem to have the "analytical mathematical-reasoning skills" still do not seem to know "how to apply these skills in a creative manner" (*ibid.*). Sternberg (1999) further asserts that "mathematics will continue to matter in the lives of most of our students, not in the test scores or course grades, but in their ability to apply the mathematics they learn to practical everyday problems" (Sternberg, 1999, p. 38). Therefore, it is crucial that "teachers give the time needed for children not only to work through activities that promote thinking but also to reflect on that thinking whenever possible" (Burns, 1985, p. 16). Furthermore, "classroom experiences must extend beyond the goal of arriving at correct answers. Children must be asked to judge the reasonableness of their thinking, to defend their solutions" (*ibid*.) and be able to express their reasoning in arriving at a mathematical solution.

Burns (1985) observes that the learners' "classroom experiences need to lead them to make predictions, formulate generalizations, justify their thinking, consider how ideas can be expanded or shifted, look for alternate approaches" (p. 16). Huscroft-D'angelo, Higgins and Crawford (2014) suggest that reasoning is a fundamental skill in mathematics hence, interventions focused on advancing student reasoning will be increasingly essential to mathematics education (p. 68).

Learners' mathematical reasoning is a broad topic that may be viewed from many perspectives. One perspective is to view learners' mathematical reasoning in terms of their reasoning processes of explanations, argumentations, justifications, and generalizations.

Explanation, argumentation, justification and generalization

Explanation, argumentation, justification and generalisation are four important elements to consider when looking for evidence of reasoning in learners. Yackel (2001) states that "students and the teacher give mathematical **explanations** to clarify aspects of their mathematical thinking that they think might not be readily apparent to others" (p. 13). Maturana & Poerksen (2004) assert that "proofs and explanations fundamentally rest on their acceptance by individuals or groups" (p. 54). Therefore, "if we accept something, we always, consciously or subconsciously, apply a criterion of validation in order to decide about the acceptability of what is to be proved and explained" (*ibid*.). Yackel (2001) advises that "the meaning of acceptable mathematical explanation is not something that can be outlined in advance for students to apply. Instead, it is formed in and through the interactions of the participants in the classroom" (p. 14).

Lithner (2000) defines **argumentation** as the "substantiation, the part of reasoning that aims at convincing oneself, or someone else, that the reasoning is appropriate" (p. 166). Burns (1985) emphasizes that "students should be encouraged to make conjectures and to examine the validity of their thoughts. They need to search for convincing arguments that support their conjectures" (p. 14). According to Chrissavgi and Vasiliki, (2013), argumentation has three general recognized forms: *analytical argumentation* – arguments that are grounded in theory of logic and proceeds inductively or deductively from a set of premises to a conclusion; *dialectical argumentation* – occurs during discussion or debate; and *rhetorical argumentation* – employed to persuade an audience (unpaged). In this study, a combination of dialectical and rhetorical argumentation will be important in my analysis of how participants use visualisation processes to solve word problems.

Justification is a practice at the heart of mathematics particularly mathematical reasoning. According to Staples, Bartlo and Thanheiser (2012), justification is used to validate claims, provide insight into a result or phenomenon, and systematize knowledge, among others purposes (p. 447). Staples et al. (2012) thus define justification as "an argument that demonstrates (or refutes) the truth of a claim that uses accepted statements and mathematical forms of reasoning" (p. 448).

Since communication is fundamental to mathematical reasoning, it makes sense that learners discuss their reasoning with others to enable them to justify their solutions (Brodie, 2010, p. 20). As a learning practice, "justification is a means by which students enhance their understanding of mathematics and their proficiency at doing mathematics; it is a means to learn and do mathematics" (Staples et al., 2012, p. 447). In addition to pursuing a demonstration of the truth of a mathematical claim, justification as a learning practice also "promotes understanding among those engaged in justification – both the individual offering a justification and the audience of that justification" (*ibid*., p. 449). Thus, the issue of identifying the strengths and limitations of visual processing in high school mathematics is an important one for mathematic educators (Presmeg, 1986, p. 42). The purpose of a justification in the context of GWPs is to provide a convincing argument for example, for justifying why carrying out a series of representations is a valid method for determining the answer of a given word problem.

To **generalise** a problem situation is to identify the operators and the sequence of operations that are common among specific cases and to extend them to the general case (Swafford & Langrall, 2000). A generalisation of a problem situation may be presented verbally or symbolically. Narrative descriptions of the general case are verbal representations of the generalisation, whereas representations using variables are symbolic representations (Swafford & Langrall, 2000). The type of generalization used for solving a problem is related to the type of the problem such that, if the relationship between the problem and the mathematical concept can be established, the reasoning type of generalization prevails (Hodnik Čadež & Manfreds Kolar, 2015, p. 283).

Generalizations as suggested by Pólya (1945, cited in Fahlgren & Brunström, 2014) are also part of the last phase in the problem solving process. His emphasis was on the learners' importance of sticking to a problem when they think that they have solved it. He encouraged that learners should utilize the opportunity to elaborate the problem further, and try to learn more from the result and the method they used. "It is instructive for students to investigate if there are related problems and if it is possible to generalize the result" (p. 291). In summary, "the literature suggests that it could be instructive for students to explore a statement further by asking ''what if…'' or ''what if not…'' questions and systematically varying key aspects to make the statement more general" (Fahlgren & Brunström, 2014, p. 291).

It is also essential to note that the product of a reasoning process is a text which warrants for a conclusion that is acceptable within the community that is producing the argument (Brodie, 2010, p. 7). "As students explain, justify and convince others of their ideas, representations are often re-examined and certain features of the representations emerge" (Sweetman, Walter, & Ilaria, 2002, p. 2). Like explanation, argumentation, justification and generalization, visualisation is a key element of mathematical reasoning. This brings us to a discussion of visualisation as a pivotal element of mathematical reasoning.

VISUALISATION

"We don't know what we see, we see what we know" – Goethe[1]

"Everything said is said by an observer" – Maturana (2004)

These are two important and inspiring statements to consider when unpacking the concept of visualisation. In his seminal panoramic interview with his friend Bernhard Poerksen, Maturana insists that "the observer is the source of everything. Without the observer, there is nothing" (Maturana & Poerksen, 2004, p. 28). Transferring this to a visualisation perspective I argue that there is no visualisation of an object unless somebody observes it. However, Arcavi (2003) cautions that "as biological and as socio-cultural beings, we are encouraged and aspire to 'see' not only what comes 'within sight', but also what we are unable to see" (pp. 215–216). Hence, "visualisation offers a method of seeing the unseen" (*ibid*., p. 216).

The importance of visualizing geometrical objects and the necessity of spatial intuition for successful mathematical teaching and learning is emphasized in the Namibian mathematics curriculum (Namibia. Ministry of Education [MoE]., 2010a). Arcavi (2003) asserts that mathematics, as a human and cultural creation dealing with objects and entities quite different from physical phenomena, relies heavily on visualisation in its different forms and at different levels, far beyond the obviously visual field of geometry, and spatial visualisation (pp. 216–217). The use of "multiple representations can be a powerful tool to facilitate students' understanding. The process of problem posing and solving that happens around the representations can foster mathematical learning" (Tripathi, 2008, p. 444). Arcavi (2003) adds that "visualisation is no longer related to the illustrative purposes only, but is also being recognized as a key component of reasoning (deeply engaging with the conceptual and not merely the perceptual), problem solving, and even proving" (p. 235).

Visualisation, which generally refers to the ability to represent, transform, generalize, communicate, document, and reflect on visual information, clearly plays a major role in the understanding of geometry (Gal & Linchevski, 2010, p. 165). Hence, "when students are translating a mathematical text into a visual representation by drawing an auxiliary figure or making a modification of a figure, they employ the strategy of visualising" (Bjuland, 2007, p. 3). In this study, I have adopted Arcavi's (2003) definition of visualisation as he defines it as follows:

> "Visualisation is the ability, the process and the product of creation, interpretation, use of and reflection upon pictures, images, diagrams, in our minds, on paper or

[1] Publication year unknown

> with technological tools, with the purpose of depicting and communicating information, thinking about and developing previously unknown ideas and advancing understanding”. (p. 217)

This definition according to Rosken and Rolka (2006) emphasizes that, visualisation can be a powerful tool in mathematics learning, that enables powerful exploration of mathematical problems and give meaning to mathematical concepts and the relationship between them. Visualisation therefore “allows for reducing complexity when dealing with a multitude of information” (p. 458).

Arcavi (2003) strongly advocates for placing visualisation at the service of problem solving as it “may play a central role to inspire a whole solution, beyond the merely procedural” (p. 224). Duval (2014) argues that visualisation “contributes to the development of imagination and creativity not only in mathematics but also in other fields of knowledge” (p. 169). Arcavi (2003) further says that “the visual display of information enables us to 'see' the story, to envision some cause-effect relationships, and possibly to remember it vividly” (p. 218). Therefore, “helping students to become aware of the importance of making drawings in mathematics problem solving” (Csíkos, Szitányi, & Kelemen, 2012, p. 62) would enable them “to engage with concepts and meanings which can be easily bypassed by the symbolic solution of the problem” (Arcavi, 2003, p. 222). Furthermore, David and Tomaz (2012) illustrate an episode “that shows how drawings of geometrical figures have a powerful role in structuring and modifying mathematical activity in the classroom” (p. 413). Gómez-Chacón (2013) on the other hand feels that other than just drawing pictures, visualisation processes in teaching requires sequenced progression of the thought process (Gómez-Chacón, 2013, p. 82).

Van Garderen, Scheuermann and Poch (2014) reveal that many different representational forms exist and many of them can be used to solve word problems. However, one often recommended strategy for solving mathematical word problems is to use visual (external) representations. Van Garderen et al. (2014) argue that a diagram can be “an extremely ‘‘powerful’’ visual representation strategy when solving word problems” *(ibid*., p. 136) hence, “diagrams as a representation strategy demonstrate great versatility as they can be used for solving various types of problems for many topic areas (e.g., geometry, number and operations, probability) and at all grade levels” (van Garderen et al., 2014, p. 136). Diagrams can also be “powerful ways to facilitate communication about critical ideas in mathematics as well as provide a platform for sharing problem solving strategies with others” (*ibid*., p. 136).

Rivera (2014) observes that “visual strategies play a mediating role in the emergence of children’s sophisticated, structured and necessary understandings of mathematical objects” (p.

59). But, van Garderen et al. (2014) caution that the ability to use a diagram as a tool for solving word problems is a complex task and should not be underestimated. Encoding information from a mathematical problem into a diagram requires an extensive knowledge base as it involves decoding the linguistic information that then is encoded into visual information. “This includes knowledge related to the ability to select, produce and productively use a diagram as a problem solving tool as well as the ability to critique and modify or generate a new diagram where needed within the context of a problem solving situation” (p. 136).

WORD PROBLEMS AND CLASSROOM CULTURE

Upon completion of the Senior Secondary Phase (SSP) in Namibia, learners are expected to be able to “use mathematical language and representation as a means of solving problems relevant to everyday life and to their further education and further careers” (Namibia. Ministry of Education [MoE], 2010, p. 23). Csíkos, Szitányi and Kelemen (2012) assert that “mathematical concepts and relations are often based on visual mental representations attached to verbal information, the ability to generate, retain and manipulate abstract images is obviously important in mathematical problem solving” (pp. 49–50). Therefore, successful problem solving requires the understanding of relevant textual information and the capacity to visualize the data (*ibid*., p. 49).

There is a need for an appropriate classroom culture to exist - a classroom culture that is clear about what constitutes desired and undesired solutions as there are many solutions to each problem but not all are acceptable in all mathematics classrooms. I argue that the socio-mathematical norms specific to mathematics needs to be explicit and clear to all the learners. For Rasmussen, Yackel and King (2006), socio-mathematical norms relate to learners' emerging beliefs and dispositions specifically related to mathematics, whilst general social norms relate to learners emerging beliefs about their own role in the classroom (p. 151). Presmeg (2014) supports the idea that the “sociocultural climate of the classroom (whether or not visualisation is accepted, encouraged and valued)” (p.152) plays an important role in how individuals tackle problem solving in mathematics. Gravemeijer (2004) presents four examples of such classroom norms:

- what counts as a mathematical problem
- what counts as a mathematical solution
- what counts as a different solution
- what counts as a more sophisticated solution (p. 6).

Gravemeijer (2004) further notes that by establishing these norms, the teacher provides the learners with criteria for judging appropriate arguments and solutions, which is very essential for the "intellectual autonomy of students" (p. 6). The learners can therefore use such criteria to make their own evaluations and do not have to wait for the final judgment of the teacher (*ibid*.). Gravemeijer (2004) thus emphasizes that, "by establishing the socio-mathematical norms; the teacher defines what mathematics is" (p. 6). Therefore, as a mathematics teacher, I argue that you do not accept any solution deemed correct for say word problems, but rather "solutions that are mathematically productive" (*ibid*., p. 7) and are mathematically correct.

Flores and Braker (2013) assert that "problems can be excellent ways to foster the development and understanding of particular mathematical concepts and procedures. However, students might use an alternative solution process that does not require the concept or process that the teacher wanted to emphasize [for example visualisation]" (p. 336). Furthermore, Flores and Braker (2013) caution that mathematics teachers need to be aware that learners might find alternative solutions strategies to particular word problems for example that do not involve their preferred problem solving strategies such as visualisation. In that case, teachers need "to decide at what point, and to what extent, they should [consider] those alternative approaches" (*ibid*., p. 336). In this study, GWPs are viewed as a means to study enacted visualisation in mathematical reasoning.

THEORETICAL PERSPECTIVE – ENACTIVISM

The observer is the source of everything. Without the observer, there is nothing – Maturana (2004)

For Maturana and Poerksen (2004), "the observer observes, sees something, affirms or denies its existence, and does whatever he does. What exists independent of him is necessarily a matter of belief and not of certain knowledge because to see something always requires someone who sees it" (p. 28). According to an enactivist perspective, "the observer is one who arises in the act of observing and whose knowing is explained through the mechanism she describes" (Simmt & Kieren, 2015, p. 307). Furthermore, the enactivist perspective asserts that reality exists in the eyes of observers and any discussion of it must begin with a description of the authors' observations (Reid, 2014, p. 138). In this study, learners are observers of the GWPs, therefore, what the learners see – they observe via their visualisations.

Enactivism "provides an inclusive, expansive, apt, and fit framework" (Khan, Francis, & Davis, 2015, p. 269) for the study of cognitive processes such as visualisation and reasoning in mathematics education. Reid (2014) employed enactivism in his study on processes of visualisation with the learning of number operations by children and reveals that "enactivism provided not only the interpretative frame but also the methodology" (p. 142). Froese (2015) echoes that enactivism provides a suitable interpretative framework for explaining the finding that emotional networks are among the most widely connected in the brain and to help us to better understand the role of "nonstandard" pathways to visual perception (p. 3). In my study enactivism will provide both the interpretive and the methodological frameworks, for example, for explaining the findings of the role that visualisation plays in the mathematical reasoning displayed by the selected learners when solving geometric word problems. As Varela (1999) asserts, objects are not seen by the visual extraction of features, but rather by the visual guidance of action (p. 14). In the context of my study this means that my research will focus on how the enacted visualisation processes enable the problem solving process (i.e. visualisation in action), as opposed to focussing only on the visualisation objects *per se*. I now unpack the aspects of enactivism that pertain to this study:

Begg (2013) defines enactivism as "a way of understanding how all organisms including human beings, organize themselves, and interact with their environments" (p. 81). It is a theory of cognition that has its roots in biological and evolutionary understandings and views human knowledge and meaning-making as processes that are understood and theorized from a biological and evolutionary standpoint (Towers & Martin, 2014, p. 249). Kieren (1994) describes enactivism as a position on cognition that includes the concepts of structural determinism, structural coupling, bringing forth a world, observer dependence, satisficing, and co-emergence. Therefore, enactivism is a name given to a "situation where we are called to position ourselves and view cognition not in terms of its products nor its mental structure, but in terms of **action** or even better, as living in the world of significance with others" (Kieren, Calvert, Reid, & Simmt, 1995, p. 2).

Fundamentally, enactivism is an effective vantage point to analyse how elements of a system work together to form that system. Maturana and Poerksen (2004) assert that living systems produce themselves within their closed dynamics and share the autopoietic organization in the molecular domain. "When we examine a living system, we find a network producing molecules that interact with others in such a way as to produce molecules that, in turn, produce the network producing molecules, and determines its boundary" (p, 98). In the context of this study, autopoiesis refers to the process of a system producing itself. It is the "self-creation [of systems] and consists of Greek words auto (self) and poiein (produce, create)" (Maturana & Poerksen, 2004, p. 97). Interactions with systems are thus crucial components of systems. Reid (2014) uses autopoiesis to describe "interactions in mathematics classrooms" (p. 140).

According to Reid (1995), an enactivist view of a problem solving situation is one in which "the person and the situation coemerge through their interaction and so the reasoning employed is both determined by the structure of the person, and occasioned by the sphere of possibilities implicit in the situation" (p. 10). The co-emergent notion that, in a problem solving situation, the problem and the problem-solver co-emerge is fundamental to this study. The actions that a problem-solver brings forth when solving a problem are thus intertwined in what the problem itself brings forth. The one cannot exist without the other. Co-emergence is thus a key theoretical concept when looking at how my participants interact with the given word problems.

Moreover, an enactivist viewpoint holds that, "one's history of interaction and one's structure determines, and at the same time is determined by, how one acts in a given setting and under various perturbations" (Simmt, 1995, p. 14). Maturana and Poerksen (2004) argue that when we are faced with new knowledge that seems to emerge out of nowhere, we create history and a domain of connections. In this way, its sudden emergence out of nowhere loses its frightening strangeness (p. 32). For example, when faced with a new and unfamiliar word problem, learners are faced with a novel situation that can be tackled from various angles. They can make use of their accumulated experiences with other previous problems and employ their repertoire of reasoning, justifications and argumentation skills to solve the new problem, or/and they can rely on an interactive process of co-emergence to come up with new and untested strategies. The point is that the problem and the problem solver constitute an intertwined system that has its own structures, and, from an enactivist perspective, these structures will determine the nature and the outcome of the problem solving process (Reid & Mgombelo, 2015, p. 173).

Damiano (2012) articulates that "co-emergence is the best notion to define the dynamical interaction between an autopoietic system and its environment, which Maturana and Varela called "structural coupling"" (p. 285). In the context of this study, the autopoietic system is the interaction between the problem and the problem solver. Damiano (2012) argues further that "it [the interaction] is a symmetric relation of reciprocal perturbations and compensations which implies the correlated emergence, in the living system and its environment, of compatible self-determined patterns of self-production" (*ibid*.). The perturbation in this study is the inherent challenges of the mathematical problem at hand.

With regard to mathematical reasoning, Reid (1995) asserts that it is essential to remain aware of the role reasoning plays in the co-emergence of learners and their situations i.e. the mathematical problems they solve. It is the structure of the learners which makes their reasoning inductively, deductively, or in some other way, possible. At the same time it is the structure of the mathematical problem in which they find themselves which occasions the reasoning they execute. Both the learner's structure and the structure of the problem, are changed by the reasoning which takes place, so that the learner and the problem coemerge through reasoning and at the same time "the reasoning is the product of that coemergence" (p. 13).

But why opt for enactivism at the expense of constructivism, say? Khan et al. (2015) write that:

> enactivist theories of human learning attend explicitly and deliberately to action, feedback, and discernment. They emphasize the bodily basis of meaning, distinguishing it from most accounts of constructivism, which, while not denying the body as ground and mediator of meaning, have not focused so intensely on the physicality of knowing and being. (p. 272)

The enactive conception of knowledge is essentially performative in contrast to constructivists that have tended to be more concerned with conceptual understanding, propositional knowledge, and webs of association (Khan et al., 2015, p. 272). While constructivism can also be interpreted as performative, "the focus is on the outcome of actions rather than the process of interactions as in enactivism" (*ibid*.).

The key concept that enactivism argues for is the inseparability of body, mind and the environment. In this study the link between the environment (the mathematical problem) and the body/mind (the problem solver) is the notion of visualisation. Visualisation is the heuristic by which the problem solving approach of the participants will be analysed. It is natural for humans to form a mental image when reading something and it is this visualisation process which is at the heart of the inseparability of the problem solver and the problem.

SIGNIFICANCE OF THE STUDY

- Embedded within the enactivist perspective, this study foregrounds the concept of visualisation in problem solving and reasoning among secondary school mathematics learners. It is therefore particularly significant in the Namibian context as it aligns well with a learner-centred approach advocated by the Namibian Curriculum. It is hoped that this study will inform teachers, researchers, curriculum designers and authors of textbooks about the significance of visualisation processes in the teaching and learning of GWPs.
- The study will contribute to the growing enactivist discourse. The use of enactivism in empirical studies is relatively novel and it is hoped that this study will enrich this discourse.
- The study will contribute towards understanding the links between visualisation, reasoning and enactivism as these are interesting and have not attracted much attention both empirically and theoretically in the context of mathematics education.
- Furthermore, my study is part of a bigger collaborative project "Visualisation in Namibia and Zambia" (VISNAMZA) which strives to promote the use of visualisation in

mathematics in Namibia and Zambia (Schäfer, 2015). Although the study is part of a bigger collaborative project, the linking of visualisation and enactivism constructs is unique in Namibia.

RESEARCH GOALS AND QUESTIONS

The aim of this study is two-fold:

1. To develop visualisation/reasoning tasks called Enacted Visualisation Geometric Reasoning Tasks (EVGRTs) that enable the examination of how visualisation processes enhance reasoning processes when solving geometry word problems.
2. To analyse how enacted visualisation processes co-emerge with mathematical reasoning during collaborative argumentations.

Hence, the study aims to answer the following research questions:

Main research question:

How does enacted visualisation enhance mathematical reasoning when solving geometry word problems?

Sub questions:

1. What visualisation processes are evident when selected Grade 11 learners solve geometry word problems?
2. How do visualisation and reasoning processes co-emerge?

METHODOLOGY

Orientation of the study

Since this study aims to examine, analyse and interpret how visualisation processes are integral to the reasoning processes when solving geometry word problems, it will be oriented within the interpretive paradigm. An interpretive paradigm according to Cohen, Manion and Morrison (2011) is characterised by a concern for the individual. The central endeavour in the context of the interpretive paradigm is to understand the subjective world of human experience. In an attempt to retain the integrity of the phenomena under study, "efforts are made to get inside the person and understand from within" (Cohen et al., 2011, p. 17).

In the context of this study I wish to understand how learners make sense of geometry word problems with specific reference to how they employ visualisation processes in their reasoning. In order to understand the learners' co-emergence of visualisation, reasoning and engagement with geometry tasks, efforts will be made to understand their world based on their experience, reflections and sense-making. From an enactivist perspective, meaning-making is not to be found in "elements belonging to the environment or in the internal dynamics of the agent, but belongs to the relational domain established between the two" (Di Paolo, De Jaegher, & Rohde, 2010, p. 40). The interpretive paradigm therefore fits an enactivist study as the interpretivists purpose to understand the meaning which informs human behaviour and to make "interpretations with the purpose of understanding human agency, behaviour, attitudes, beliefs and perceptions" (Bertram & Christiansen, 2014, p. 26) aligns well with the enactivist notion of co-emergence.

Method

This research takes the form of a qualitative case study. Qualitative research aims to understand the meaning of phenomena under study as well as the relationships among naturally occurring variables (Ross & Onwuegbuzie, 2012, p. 86). Cohen et al. (2011) define a case study research as a specific and a single instance of a bounded system such as a child, a class, or a school, frequently designed to illustrate a phenomenon in a particular context. Bell (1993) believes that "a case study approach is particularly appropriate for individual researchers because it gives an opportunity for one aspect of a problem to be studied in some depth within a limited time scale" (p. 8). Cohen et al. (2011) suggest that a good case study research requires in-depth data, a researcher with the ability of gathering data that address fitness for purpose, and skilled in probing beneath the surface of phenomena. This implies that a case study researcher "must be an effective questioner, listener (through many sources), prober, able to make informed inferences and adaptable to changing and emerging situations" (p. 296).

From an enactivist perspective, attention is given to the "on-going co-constructed interaction among bodily actions [using various visualisation processes], cognition and the environment [the word problems]" (Khan et al., 2015, p. 272). These permit the co-emergence of the participant (learner) and his/her environment (word problems) through enacted learning (visualisation). During the second phase of data analysis, close attention will be given to the relationship between visualisation and reasoning, and how these two processes co-emerge. My case will thus be a cohort of Grade 11 learners engaging with geometry word problems, whilst the units of analysis are the observed visualisation and reasoning processes of the selected learners as they solve these problems in a classroom setting.

Site and sample

In this study, I will make use of a non-probability, purposeful sampling method (Cohen et al., 2011) to select my participants. I will select a cohort of 20 mixed gender and mixed ability Grade 11 learners in consultation with their mathematics teachers. Grade 11 learners will be selected for this study because they are familiar with different concepts of geometry, and still have time to participate in this study, as opposed to 12 learners whose focus is dominated by the end-of-year national examinations. The sample is purposeful in the sense that I need 20 learners that are able to converse well, that can articulate clearly and who can express themselves richly. The involvement of teacher colleagues in the selection of these participants is thus crucial. The reason why I opted for a mixed ability group is that the use of visualisation processes is generic and not specific to a particular ability group.

The research site (the school where the 20 participating learners are located) will be conveniently selected on the basis of its availability and accessibility (Cohen et al., 2011). I have a good relationship with the principal and the mathematics teachers at the school which will facilitate a smooth recruiting and buy-in process of the participants and their parents.

Enacted Visualisation Geometric Reasoning Tasks (EVGRTs)

My central instruments in the research process are the Enacted Visualisation Geometric Reasoning Tasks (EVGRTs) Worksheets 1 and 2. The EVGRTs Worksheets will each consist of 10 geometry word problems that will be sourced and developed in alignment with the Grade 11 syllabus. It is one of the aims of this study to develop the EVGRTs. The tasks for the two EVGRTs worksheets will initially be sourced from textbooks and other resources. They will then be adapted to suit the needs of this study; i.e. reflect a Namibian context and provoke visualisation processes in their engagement.

Both EVGRTs Worksheet 1 and 2 will each consist of 10 tasks that promote the use of visualisation processes to solve them. These will be tasks where the learners will invariably have to sketch or draw something. They will be requested to refer to their sketches for discussions and reflections. Shulman (2004) supports that "the more complex and higher-order the learning, the more it depends on reflection—looking back—and collaboration—working with others." (p. 319)". The participants will engage with the tasks as per the two phases outlined below. The interactions will be video recorded in order to observe and analyse their reasoning and visualisation processes as they solve each task. Learners will solve Worksheet 1 individually and Worksheet 2 in small-groups as explained below.

The following are two examples of two tasks of the EVGRTs Worksheets:

EVGRTs worksheet 1: Work Individually

Instruction: Read the following problems carefully and answer them by showing your work.

Task 1:

Town C is located due east of town A. Town B is located due north of town A and 20 km from town C. If town B is on a bearing of 330° from town C. How far east of town A is town C?

Figure 1: An example of one of the tasks of EVGRTs Worksheet 1

EVGRTs worksheet 2: Work in small groups

Instruction: Study the following problems in your group and give a collective solution, showing all your work.

Task 1:

If town A and town B are eight kilometres apart and town C is ten kilometres from B, what are the possible distances from town A to town C?

Figure 2: An example of one of the tasks of EVGRTs Worksheet 2

DATA COLLECTION

The research design is divided into two phases:

PHASE 1 – EVGRTs Worksheet 1

This phase comprises of the participants' selection, the development and the initial administering of the EVGRTs worksheet 1. Before administering the EVGRTs worksheet 1 to the 20 selected learners it will be piloted with a group of Grade 11 learners who are not research participants. The purpose of this pilot will be to identify and correct ambiguities and inaccuracies in the tasks in order to ensure reliability and validity of the EVGRTs worksheet. After piloting, each of the 20 selected participants will solve all the 10 tasks individually with me. I will ask each participant to talk through each of the 10 items with me while solving them. I will specifically focus on the nature of the visualisation strategies that each participant employs and ask for elaboration and clarification on how the visualisation processes assist in solving the tasks. As the tasks are only written in English, I will assist those learners whose home language is not English and who do not understand the question, by translating the task into the learners' home language. Data from this phase consists of the participants' pencil and paper solutions, my field notes, as well as the transcribed video recordings of the talk-through sessions.

The purpose of Phase 1 is twofold. Firstly, to tease out how the learners solve EVGRTs in terms of visualisation processes. Secondly, to categorise the learners who favoured visual strategies over numeric or algebraic strategies when solving geometry word problems (visualisers) from the nonvisualisers. Presmeg (1986b) differentiates that "visualisers are individuals who prefer to use visual methods when attempting mathematical problems which may be solved by both visual and nonvisual methods. Nonvisualisers are individuals who prefer not to use visual methods when attempting such problems" (p. 42). The visually oriented participants will then be selected from the 20 participants to form the cohort for data collection phase 2. I anticipate not more than 10 learners to form this cohort.

PHASE 2 – EVGRTs Worksheet 2

The selected cohort of 10 participants from the first phase will be split into small groups of 2 - 3 learners to enable collaborative argumentation whilst solving EVGRTs worksheet 2. According to Prusak, Hershkowitz and Schwarz (2012), a collaborative argument refers to a “situation in which two or more people learn or attempt to learn something together” (p. 23). The specific purpose of this phase is to investigate how enacted visualisation enhances mathematical reasoning and co-emerge with each other. Data for this phase will be collected in three stages. The first stage consists of observing how members of each small group collaboratively engaged with each other when solving each of the 10 tasks. The focus will be to look for evidence of argumentation and reasoning. The purpose of this stage is specifically to examine the emergence of mathematical reasoning through enacted visualisation processes when solving geometry word problems. Data from this stage consists of video recordings of each group. The second stage comprises of interviews with each of the small groups. The aim of interviews is to reflect on how each of the 10 EVGRTs was solved during stage 1. The third stage is a focus group interview with the whole cohort. The purpose of this stage is to have an overall reflection of the entire problem solving process. The last two stages will be audio recorded.

DATA ANALYSIS

Data will be analysed in two phases as it is generated.

DATA ANALYSIS PHASE 1

Data for this phase will be analysed to determine which participants prefer visualisation strategies over other strategies when solving GWPs. The participants’ explanations and justifications in using visualisation strategies will be thoroughly analysed to identify the cohort that would proceed to the next phase of data collection. Visualisation processes will be classified according to Presmeg’s (1986b) categories of visual imagery: *concrete imagery*, *pattern imagery*, *memory images of formulae*, *kinaesthetic imagery* and *dynamic imagery*. The table below represents the analytical instrument of the visualisation categories and observable indicators that will be used to analyse the participants’ preference of solving geometry word problems. Presmeg’s categories will be used to code the data for analysis purposes.

Category	**Definition**	**Observable Indicators** **The learner provides evidence of:**	**Descriptive Frequency**
Concrete, pictorial imagery	Concrete image of an actual situation formulated in a person’s mind; picture in the mind drawn on paper or described verbally	✓ Drawing pictures to represent a concrete situation. ✓ Formulating a picture in the mind while reading a word problem.	
Pattern imagery	This is the generalisation of known problem solving strategies	✓ Making use of known patterns when drawing/sketching.	

Memory images of formulae	This refers to the ability to visualise an image that one has seen somewhere before.	✓ Visualising a picture of a formula in his/her mind; i.e. articulating/describing his/her picture. ✓ Visualising a book or a chalkboard and depict how the formula was written when he/she saw it.	
Kinaesthetic imagery	This is imagery that involves muscular activity.	✓ Using his/her hands/fingers to indicate a path on drawn images ✓ Making gestural indications of particular geometric shapes.	
Dynamic imagery	This category involves the processes of redrawing given or initially own drawn figures with the aim of solving the problem.	✓ Redrawing given or own drawn/sketched figures for the purpose of extracting simple figures i.e. extracting a right angled triangle from a complex 3-D shape.	

Table 1: Presmeg's categories of visualisation processes for determining participants who favour visual modes of problem solving

The purpose of using the above categories is mainly to answer research sub-question one which seeks to identify the visualisation processes that are evident when selected Grade 11 learners solve geometry word problems. By employing the indicators of Table 1, I will identify those learners who mostly prefer a visual as opposed to a non-visual mode of solving GWPs, as well as the dominant visualisation processes that they employ during problem solving. I propose to count the frequency of each category for each of the 20 participants that will answer each of the 10 EVGRTs worksheet 1 to determine the dominant visualisation processes. Data for this phase will be analysed from the one-on-one task based interviews when I talk-through each participants' problem solving process. I will be looking for evidence of visualisation in individual participants' problem solving strategies as explained. Twenty of these tables will be completed and thereafter discussed and analysed to determine the participants' preferred mode of solving word problems. A short summary of each table will help to select the cohort for the second phase of data collection.

During this phase, I will engage a critical friend to verify and validate my analysis of the tasks in terms of classifying the participants into visualisers and nonvisualisers. The critical friend will do two things: one is to validate my analysis and the other is to validate the sample for phase 2.

DATA ANALYSIS PHASE 2

Data for this phase will be analysed in three stages.

Stage 1 – video analysis of small group collaborative argumentation

The focus of my analysis of the video recordings of each group interacting with the EVGRTs worksheet 2 will be on how the participants argue and reason in the context of using visualisation processes when solving the 10 tasks of the worksheet. Table 2 below shows my analytical tool with indicators which will enable me to analyse this data.

	Explanation	Argumentation	Justifications	Generalisation
Reasoning Processes	**The learner clarifies aspects of mathematical thinking that might not be evident to others.**	**The learner convinces others of his/her understanding/explanations of mathematical concepts and ideas.**	**The learner validates his/her claims, provides insight into a phenomenon, and promotes understanding.**	**The learner identifies common operators and the sequence of operation to extend them to the general case.**
Indicators	The learner(s) provide evidence of: • insight and elaborations into why a statement is true • interacting in the classroom • clarifying concepts and ideas • explaining his/her thinking to others, asking and raising questions and challenges other. • using drawings to explain and illustrate mathematical concepts and processes	The learner(s) provide evidence of: • example-based reasoning • engaging in dialogue • critical thinking • making decisions and performing tasks • interacting and inter-thinking • providing reasons for each claim • reasoning through drawings, pictures and general images • convincing and persuading others of his/her understanding of concepts and ideas • constructing arguments • validating claims	The learner(s) provide evidence of: • meaning-making • engaging with structure and meaning of GWPs • clarifying own thinking • validating claims • providing insight into a phenomenon • enhancing understanding of word problems and proficiency at solving them • listening to and making sense others' explanations and claims	The learner(s) provide evidence of: • introducing new ideal objects • connecting to previous theorems or axioms to solve current problems • relating type of generalisation to type of problem • recognising additional similar experience when solving new problems • discovering principles and making necessary connections of the phenomena within a certain whole • solving related problems by applying schemas

Table 2: My analytical framework of observable indicators of reasoning processes (using the works Yackel (2001), Chrissavgi and Vasiliki (2013), Staples et al. (2012) and Hodnik Čadež and Manfreds Kolar (2015). This is a work in progress and needs further refining.

Stage 2 – small-groups interview analysis

In this stage I wish to analyse the interviews of each small group in order to gain deeper insight into how the participants' visualisation processes co-emerged with their reasoning processes. I will be seeking emerging themes as a result of the participants' experiences with the EVGRTs worksheets and collaborative argumentations. I am aware of the danger of probing questions during interviews as they often lead to leading questions. Therefore, I will minimise this danger by going into the interview sessions with prepared questions such as: What are you imagining now? What is going through your mind? Is there a picture in your mind? Can you draw the picture for me? What does that picture mean to you?

Stage 3 – focus group interview analysis

As this interview consists mainly of reflections and consolidation about the entire research process and journey, this stage will be analysed for dominant themes, perceptions and experiences.

VALIDITY

By implication, validity is essential to verify and conclude the authenticity of generated data. Cohen et al. (2011) propose that “in qualitative data validity might be addressed through honesty, depth, richness and scope of the data achieved ... [and] might be improved through careful sampling, appropriate instrumentation and appropriate statistical treatments of the data” (p.179). Pietersen and Maree (2011) inform that “the validity of an instrument refers to the extent to which it measures what it is supposed to measure” (p. 216).

There will be two levels of piloting: the focus of Pilot 1 will be on the developed EVGRTs Worksheets. The aim of this pilot is to refine the instrument once I have selected the 10 items for each worksheet for language, mathematical accuracy and appropriateness. The focus of Pilot 2 will be on the analytical instruments which I will use to analyse my data of the two Phases. Pilot 2 will assist me in refining the observable criteria and indicators by making sure that they are clear and non-ambiguous. The refinement will probably happen throughout the course of the analysis. This means that I will pilot a full cycle of data collection with two or three Grade 11 learners that are not part of my cohort of participants before finalising my instruments.

ETHICAL CONSIDERATIONS

See attached form of Rhodes University ethical considerations for the Faculty of Education.

PROPOSED RESEARCH SCHEDULE

Proposed Schedule 2015	Jan	Feb	Mar	Apr	May	Jun	Jul	Aug	Sep	Oct	Nov	Dec
Literature review and study												
Proposal development												

Proposed Schedule 2016	Jan	Feb	Mar	Apr	May	Jun	Jul	Aug	Sep	Oct	Nov	Dec
Proposal development												
Piloting												
Phase 1 data collection												
Phase 1 data analysis												
Phase 2 data collection												
Phase 2 data analysis												

Formal write-up												

Proposed Schedule 2017	Jan	Feb	Mar	Apr	May	Jun	Jul	Aug	Sep	Oct	Nov	Dec
Formal write-up												
Final proofing and submission												

REFERENCES

Ahmad, A., Tarmizi, R. A., & Nawawi, M. (2010). Visual representations in mathematical word problem solving among form four students in Malacca. In *Proceedings of the International Conference on Mathematics Education Research* (pp. 356–361). http://doi.org/10.1016/j.sbspro.2010.12.050

Arcavi, A. (2003). The role of visual representations in the learning of mathematics. *Educational Studies in Mathematics*, *52*(3), 215–241.

Begg, A. (2013). Interpreting enactivism for learning and teaching. *Education Sciences & Society*, 81–96.

Bell, J. (1993). *Doing your research project: A guide for first-time researchers in education and social science* (2nd ed.). Buckingham: Open University Press.

Bertram, C., & Christiansen, I. (2014). *Understanding research: An introduction to reading research*. Pretoria: Van Schaik Publishers.

Bjuland, R. (2007). Adult students' reasoning in geometry: Teaching mathematics through collaborative problem solving in teacher education. *Montana Mathematics Enthusiast*, *4*(1), 1–30. Retrieved from http://search.ebscohost.com/login.aspx?direct=true&db=a9h&AN=24277217&lang=es&site=ehost-live

Boesen, J., Lithner, J., & Palm, T. (2010). The relation between types of assessment tasks and the mathematical reasoning students use. *Educational Studies in Mathematics*. http://doi.org/10.1007/s10649-010-9242-9

Brodie, K. (2010). *Teaching mathematical reasoning in secondary school classrooms*. New York: Springer.

Burns, M. (1985). The Role of questioning. *Source: The Arithmetic Teacher*, *32*(6), 14–16. Retrieved from http://www.jstor.org

Chen, C. L., & Herbst, P. (2013). The interplay among gestures, discourse, and diagrams in students' geometrical reasoning. *Educational Studies in Mathematics*. http://doi.org/10.1007/s10649-012-9454-2

Chrissavgi, T., & Vasiliki, S. (2013). Periodicity in mathematics and science texts: The co-deployment of visual representation. Retrieved June 22, 2015, from www.esera.org/media/eBook-2013/strand 7/Chrissavgi-Triantafillou_15Dec2013.pdf

Cohen, L., Manion, L., & Morrison, K. (2011). *Research methods in education* (7th ed.).

London: Routledge/Taylor & Francis Group.

Csíkos, C., Szitányi, J., & Kelemen, R. (2012). The effects of using drawings in developing young children's mathematical word problem solving: A design experiment with third-grade Hungarian students. *Educational Studies in Mathematics*. http://doi.org/10.1007/s10649-011-9360-z

Damiano, L. (2012). Co-emergences in life and science : a double proposal for biological emergentism. *Synthese*, *185*, 273–294. http://doi.org/10.1007/s11229-010-9725-3

David, M. M., & Tomaz, V. S. (2012). The role of visual representations for structuring classroom mathematical activity. *Educational Studies in Mathematics*, *80*(3), 413–431. http://doi.org/10.1007/s10649-011-9358-6

Dejarnette, A. F., & González, G. (2013). Building students' seasoning skills by promoting student-led discussions in an algebra II class. *The Mathematics Educator*, *23*(1), 3–23.

Di Paolo, E., De Jaegher, H., & Rohde, M. (2010). Horizons for the enactive nind: values, social interaction, and play. In J. Steward, O. Gapenne, & E. Di Paolo (Eds.), *In enaction: Towards a new paradigm for cognitive science.* (pp. 31–87). Cambridge, MA: MIT Press. http://doi.org/10.1098/rsif.2004.0012

Duval, R. (2014). Commentary: Linking epistemology and semio-cognitive modeling in visualization. *ZDM - Mathematics Education*, *46*, 159–170. http://doi.org/10.1007/s11858-013-0565-8

English, L. D. (1999). Reasoning by analogy: A fundamenta process in children's mathematical learning. In L. V. Stiff & F. R. Curcio (Eds.), *Developing mathematical reasoning in Grades K-12* (pp. 22–36). Reston: NCTM.

Fahlgren, M., & Brunström, M. (2014). A model for task design with focus on exploration, explanation, and generalization in a dynamic geometry environment. *Technology, Knowledge and Learning*, *19*(3), 287–315. http://doi.org/10.1007/s10758-014-9213-9

Flores, A., & Braker, J. (2013). Developing the art of seeing the easy when solving problems. *The Mathematics Enthusiast*, *10*(1&2), 365–378.

Froese, T. (2015). Enactive neuroscience, the direct perception hypothesis, and the socially extended mind 1. *Behavioral and Brain Sciences*, *(In Press)*, 1–6.

Gal, H., & Linchevski, L. (2010). perception in geometry To see or not to see : analyzing difficulties from the perspective of visual perception. *Educational Studies in Mathematics*, *74*(2), 163–183. http://doi.org/10.1007/sl0649-010-9232-y

Gómez-Chacón, I. M. (2013). Prospective teachers ' interactive visualization and affect in mathematical problem-solving. *The Mathematics Enthusiast*, *10*(1&2), 61–86.

Gravemeijer, K. (2004). Creating opportunities for students to reinvent mathematics. In *Proceedings of the 10th Conference of the International Congress of Mathematics Education* (pp. 1–17).

Hodnik Čadež, T., & Manfreds Kolar, V. (2015). Comparison of types of generalizations and problem-solving schemas used to solve a mathematical problem. *Educational Studies in*

Mathematics, *89*, 283–306. http://doi.org/10.1007/s10649-015-9598-y

Huscroft-D'angelo, J., Higgins, K., & Crawford, L. (2014). Communicating mathematical ideas in digital writing environment: The impacts on mathematical reasoning for students with and without learning disabilities. *Social Welfare Interdisciplinary Approach*, *4*(2), 68–84.

Jao, L. (2013). From Sailing Ships to Subtraction Symbols: Multiple Representations to Supp...: Discovery Service for Rhodes University Library. Retrieved June 26, 2015, from http://0-eds.a.ebscohost.com.wam.seals.ac.za/eds/detail/detail?sid=84da5b9f-b074-409b-92d9-1adfb305d235@sessionmgr4002&crlhashurl=login.aspx%3fdirect%3dtrue%26site%3deds-live%26db%3deric%26AN%3dEJ1025587&hid=4108&vid=0&bdata=JnNpdGU9ZWRz

Khan, S., Francis, K., & Davis, B. (2015). Accumulation of experience in a vast number of cases: enactivism as a fit framework for the study of spatial reasoning in mathematics education. *ZDM*, *47*, 269–279. http://doi.org/10.1007/s11858-014-0623-x

Kieren, T. E., Calvert, L. G., Reid, D. A., & Simmt, E. (1995). Coemergence: Four enactive portraits of mathematical activity. In *Paper presented at the Annual Meeting of the American Educational Research Association* (pp. 1–35).

Lithner, J. (2000). Mathematical Reasoning in school tasks. *Educational Studies in Mathematics*. http://doi.org/10.1023/A:1003956417456

Malloy, C. E. (1999). Developing mathematical reasoning in the middle grades: Recognizing diversity. In L. V. Stiff & F. R. Curcios (Eds.), *Developing mathematical reasoning in Grades K-12* (pp. 13–21). Reston: NCTM.

Maturana, H. R., & Poerksen, B. (2004). *From being to doing: The origin of the Biology of cognition*. Heidelberg: Carl-Auer Verlag.

Mueller, M., Yankelewitz, D., & Maher, C. (2014). Teachers promoting student mathematical reasoning. *The Research Council on Mathematics Learning*, *7*(2), 1–20.

Namibia. Ministry of Education [MoE]. (2010a). The national curriculum for basic education. Okahandja: NIED.

Namibia. Ministry of Education [MoE]. (2010b). *The National Curriculum for Basic Education*. Okahandja.

Pietersen, J., & Maree, K. (2011). Standardization of a questionnaire. In K. Maree (Ed.), *First steps in research* (pp. 215–222). Pretoria: Van Schaik Publishers.

Presmeg, N. (2014). Contemplating visualization as an epistemological learning tool in mathematics. *ZDM - International Journal on Mathematics Education*, *46*, 151–157. http://doi.org/10.1007/s11858-013-0561-z

Presmeg, N. C. (1986a). Visualisation and mathematical giftedness. *Educational Studies in Mathematics*, *17*(3), 297–311. http://doi.org/10.1007/BF00305075

Presmeg, N. C. (1986b). Visualisation in high school mathematics. *For the Learning of Mathematics*, *6*(3), 42–46.

Prusak, N., Hershkowitz, R., & Schwarz, B. B. (2012). From visual reasoning to logical necessity through argumentative design. *Educational Studies in Mathematics*, *79*(1), 19–40. http://doi.org/10.1007/s10649-011-9335-0

Rasmussen, C., Yackel, E., & King, K. (2006). Social and sociomathematical norms in the mathematics classroom. In H. L. Schoen & R. I. Charles (Eds.), *Teaching mathematics through problem solving* (pp. 143–154). Reston: NCTM.

Reid, D. A. (1995). A coemergence of reasoning. In T. E. Kieren, L. G. Calvert, D. A. Reid, & E. S. M. Simmt (Eds.), *Paper presented at the Annual Meeting of the American Educational Research Association* (pp. 10–13).

Reid, D. A. (2014). The coherence of enactivism and mathematics education research : A case study. *AVANT*, *V*(2), 137–172. http://doi.org/10.12849/50202014.0109.0007

Reid, D. A., & Mgombelo, J. (2015). Survey of key concepts in enactivist theory and methodology. *ZDM - Mathematics Education*, *47*, 171–183. http://doi.org/10.1007/s11858-014-0634-7

Rivera, F. D. (2014). From math drawings to algorithms: Emergence of whole number operations in children. *ZDM - International Journal on Mathematics Education*, *46*, 59–77. http://doi.org/10.1007/s11858-013-0543-1

Rosken, B., & Rolka, K. (2006). A picture is worth a 1000 words – The role of visualisation in mathematics learning. In *Proceedings of the 30th Conference of the International Group for the Psychology of Mathematics Education* (pp. 457–464).

Ross, A., & Onwuegbuzie, A. J. (2012). Prevalence of Mixed Methods Research in Mathematics Education. *The Mathematics Educator*, *22*(1), 84–113.

Russell, S. J. (1999). Mathematical reasoning in the elementary grades. In L. V. Stiff & F. R. Curcio (Eds.), *Developing mathematical reasoning in Grades K-12* (pp. 1–12). Reston: NCTM.

Schäfer, M. (2015). *Visualisation in Namibia and Zambia project 2015. Concept document*. Grahamstown.

Shulman, L. S. (2004). Teaching alone, learning together: Needed agendas for the new reform. In L. S. Shulman (Ed.), *The wisdom of practice: Essays on teaching, learning, and learning to teach* (pp. 309–333). San Francisco, CA: Jossey-Bass.

Simmt, E. (1995). A portrait of beliefs is action. In T. E. Kieren, L. G. Calvert, D. A. Reid, & E. Simmt (Eds.), *Paper presented at the Annual Meeting of the American Educational Research Association* (pp. 14–18).

Simmt, E., & Kieren, T. (2015). Three “Moves” in enactivist research: a reflection. *ZDM*. http://doi.org/10.1007/s11858-015-0680-9

Staples, M. E., Bartlo, J., & Thanheiser, E. (2012). Justification as a teaching and learning practice: Its (potential) multifacted role in middle grades mathematics classrooms. *Journal of Mathematical Behavior*, *31*(4), 447–462. http://doi.org/10.1016/j.jmathb.2012.07.001

Stein, M. K., Grover, B. W., & Henningsen, M. (1996). Building student capacity for mathematical thinking and reasoning: An analysis of mathematical tasks used in reform classrooms. *American Educational Research Journal*, *33*(2), 455–488. http://doi.org/10.3102/00028312033002455

Sternberg, R. J. (1999). The nature of mathematical reasoning. In L. V Stiff & F. R. Curcio (Eds.), *Developing mathematical reasoning in Grades K-12* (pp. 37–44). Reston.

Sweetman, T. D., Walter, J. G., & Ilaria, D. R. (2002). Understanding , justification , and representation : Secondary students and emergent strands in mathematics education case study literature. In *Paper presented at the Annual Meeting of the American Educational Research Association* (pp. 2–13).

Towers, J., & Martin, L. C. (2014). Enactivism and the study of collectivity. *ZDM Mathematics Education*, *47*, 247–256. http://doi.org/10.1007/s11858-014-0643-6

Tripathi, P. N. (2008). Developing mathematical understanding through multiple representations. *Mathematics Teaching in the Middle School*, *13*(8), 438–445. Retrieved from http://www.jstor.org/stable/41182592

van Garderen, D., Scheuermann, A., & Poch, A. (2014). Challenges students identified with a learning disability and as high-achieving experience when using diagrams as a visualization tool to solve mathematics word problems. *ZDM - International Journal on Mathematics Education*, *46*, 135–149. http://doi.org/10.1007/s11858-013-0519-1

Varela, F. J. (1999). *Ethical know-how: Action, wisdom and cognition*. (T. Lenoir & H. U. Gumbrecht, Eds.). Stanford: Stanford University Press.

Yackel, E. (2001). Explanation, justification and argumentation in mathematics classrooms. In *Proceedings of the 25th Conference of the International Group for the Psychology of Mathematics Education* (pp. 9–24).

Zimmermann, W., & Cunningham, S. (1991). Editors' introduction: What is mathematical visualization? In W. Zimmermann & S. Cunningham (Eds.), *Visualization in teaching and learning mathematics* (pp. 1–8). Washington: Mathematical Association of America.

RU FACULTY OF EDUCATION: ETHICAL APPROVAL APPLICATION

Med (Half thesis)		**Med Full thesis)**		**PhD**	**X**	**Others: Please specify**	

TITLE OF RESEARCH: Examining mathematical reasoning through enacted visualisation when solving word problems.

DEPARTMENT/INSTITUTE: Education Department
DATE: 12 January 2016
RESEARCHER: Beata Lididimikeni Dongwi
SUPERVISOR: Professor Marc Schäfer

ETHICS

Respect and Dignity

The participants will be informed of all the implications of their voluntary participation and their freedom to withdraw without notice. Their dignity, privacy and interests will be respected and protected at all times. The participants' consent will be formally sought and the data will remain confidential between my supervisor and me. The findings of the research will be made available to participants upon request. All the planned research activities will take place after formal school activities and when it is most convenient for each participant. I will ensure that there will be mutual respect throughout the research process.

Transparency and honesty

Written informed consent will be sought from all stakeholders in the research process. The purpose and the procedures of the research will be clearly explained to all participants. I will be transparent and clear about the criteria for selecting the research participants. I will consult closely with the Grade 11 mathematics teachers to identify learners who are able to converse well, who can articulate clearly and who can express themselves richly. Should an incentive, such as refreshments, be awarded, I will ensure that it does not become a bribe by justifying my reasons explicitly.

Accountability and Responsibility

I am aware of my position as a mathematics teacher and I will reassure the participants that my position should not compromise their responses. I will make it clear to the participants that the purpose of the EVGRTs is not for testing their mathematical proficiency – hence I will not judge their performance. I will reassure the participants that I am interested in their process of tackling the 10 GWPs and not necessarily their product. I am also aware of the dangers of leading questions during the task-based interviews. Therefore, I will try to minimise this danger by preparing my probing questions in advance and leaving sufficient time and space for my participants to express themselves fully. The EVGRTs Worksheets are not time trials, and the participants will be encouraged to use as much time as they need to solve the tasks.

Integrity and Academic professionalism

I will ensure that my research design is transparent, clearly articulated and robust in order to safeguard my integrity not to be accused of data manipulation. The research will be my own work, written in my own words and where I draw on others works or ideas, I will appropriately acknowledge and reference them.

___B.Dongwi__________________

Signature (researcher) Signature (supervisor)

Date: 13 January 2014 Place: Okahandja and Grahamstown